THE NOTION OF A CONTINUUM.[1]

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BY a continuum is understood a system or manifoldness of parts possessed in varying degree of a property A, such that between any two parts distant a finite length from each other, an infinite number of other parts may be interpolated, of which those that are immediately adjacent exhibit only infinitely small differences with respect to the property A.

There can be no objection to such a system, considered as a fiction merely, or as a purely arbitrary ideal construct. But the natural inquirer, who is not exclusively concerned with the purely mathematical point of view, is compelled to inquire whether there is anything in *nature* that corresponds to such a fiction. Space viewed in its simplest form as a succession of points in a straight line, time viewed as the succession of the elements of a uniformly sounding musical note, the succession of colors shown by the spectrum with the Fraunhofer lines obscured, are typical instances of the kind of continua presented in nature. If we consider such a "continuum" solely in the light of facts, it will be seen that there is nothing perceptible by the senses corresponding to an infinite number of parts or to infinitely minute differences. All we may say is, that in traversing such a succession, the differences between the parts increase as the parts move away from each other, until ultimately these differences admit of not the slightest doubt; and again, that as the parts approach each other the differences decrease, that afterwards it is alternately possible and impossible to distinguish them, according to chance and circumstances, and that finally it is altogether impossible to do so. *Points* of space and time do not exist for sense-perception; there exist for such, only spaces and times so small as not to admit of more minute division percep-

[1] Translated from the *Wärmelehre* by T. J. McCormack.

tible to the senses, or so small that we consciously neglect their size, although on increased attention they might admit of resolution into component elements. The possibility of passing imperceptibly and uninterruptedly from a property A to a property A', sharply distinguishable from A, is the important point. The fact is, that any two terms on given trial are either distinguishable or undistinguishable.

It is possible to remove a large number of parts from a given sensory continuum without causing the system to cease giving the impression of a continuum. If we imagine a large number of narrow equidistant bands of color cut out of a spectrum, and the remainder pushed together until the parts touch, the spectrum will still give the impression of a color-continuum, in spite of the interruption of continuity in the wave-lengths of the lines. In like manner, an ascending musical note, if the intervals between the rates of vibration be sufficiently small, may be regarded as a continuum, and the jolting movement produced by a sufficiently large number of successive but detached stroboscopic pictures may also be made to appear as a continuous movement.

If the parts of a sensory continuum stood forth as individual entities and were distinguishable with absolute accuracy, the employment of artificial expedients, as the use of measures for comparing continua of the same kind and the use of dividing lines for rendering imperceptible differences of space distinct by means of conspicuous differences in color, etc., would be superfluous. But the moment we introduce such artifices as being superior physically for the indication of the differences, we abandon the domain of immediate sense-perception, and pursue a course in every respect similar to that of substituting the thermometer for the sensation of heat. A distance in which the measure is contained twice or three times, is then twice or three times that in which it is contained once; and the hundredth part of the measure corresponds to a hundredth part of the difference, although it may not be said that this difference holds good for direct perception. With the introduction of the measure, a new definition of distance or difference has been introduced. Judgments of difference are now no longer formed from simple sense-perception, but are reached by the more complex reaction involved in the application of the measure; and the result depends upon the issue of the experimental test. The consideration last adduced may be profitably called to the attention of that still large body of thinkers who refuse to admit that the axioms of geometry are the results of experience,—results *not* given by direct perception when *metrical* concepts are introduced.

The employment of measures suggests the employment of numbers, but the use of the latter is not necessarily entailed until it is resolved to employ only one measure, which is multiplied or subdivided according as the necessity arises for a larger or smaller continuum of comparison. In using a measure divided into absolutely equal parts, we are immediately enabled to employ all the numeral experiences which we have gained from our study of discrete objects. This is not the place for a detailed discussion of the manner in which operations of counting themselves gave rise to the necessity of new numeral concepts far transcending the bounds of the original system of integer positive numbers and of the gradual manner in which negative and fractional numbers, and finally the entire system of rational numbers, came into being.

If a unit is to be divided, it must either exhibit natural parts for such a division, as for example do many fruits, or it must at least permit of being conceived as made up of perfectly homogeneous equivalent parts. The early appearance of unit-fractions is a probable indication that division was learned by experiences of the first-mentioned kind, and that the skill acquired in that field was carried over to cases of the second class, namely, to the division of continua. It is here apparent from the simplest instances that the number-system which originated from the consideration of discrete objects is inadequate for the representation of fluent or continuous states. For instance, the common fraction $\frac{1}{3}=0.333333\ldots$ A point of trisection, in other words, can never be found exactly by decimal subdivision, however minute. The ratios of certain line-segments, as that of the diagonal to the side of the square, are absolutely unrepresentable by rational numbers, as Pythagoras long ago discovered,[1] and lead immediately to the concept of the irrational.[2]

The cases of this are innumerable. It may be expressed by saying that "the straight line is infinitely richer in point-individuals than the domain of rational members is in number-individuals."[3] But the remark is applicable, as the illustration given above of the

[1] Euclid's ingenious proof of this proposition is found in his *Elements*, X, 117. Compare Cantor's views in his *Geschichte der Mathematik*, pp. 154, et seq.

[2] The irrational number $\sqrt{p}$ is the limit between all rational numbers (1) the squares of which are less and (2) the squares of which are greater than p. In the first class no greatest, and in the second no least, number can be assigned. If $\sqrt{p}$ is rational, the number in question is the greatest of the first and the least of the second class. Compare Tannery, *Théorie des Fonctions*, Paris, 1886

[3] Dedekind, *Stetigkeit und irrationale Zahlen*, Brunswick, 1892.

point of trisection shows, quite irrespective of the irrational feature, to every *special* number-system. We might say $\frac{1}{3}$ is a relative irrational number, as compared with the decimal system.

Numbers, which were originally created for the intellectual mastery of discrete objects, accordingly prove themselves to be absolutely inadequate for the mastery of continua which are conceived as inexhaustible, be these real or fictitious. Zeno's assertion of the impossibility of motion on account of the infinite number of the points that had to be traversed between the initial and terminal stations, was admirably refuted in this sense by Aristotle, who remarked that "a moving object does not move by numbers."[1] The idea that we are obliged to exhaust all things by counting is due to the inappropriate employment of a method which, for a great many cases, is quite appropriate. A pathological phenomenon of what might be called the counting-mania actually makes its appearance here. No one will be inclined to discover a problem in the fact that the series of natural numbers can be continued upwards as far as we please, and consequently can never be completed; and it is not a whit more necessary to discover a problem in the fact that the division of a number into smaller and smaller parts can be continued *ad libitum* and consequently never completed.

At the time of the founding of the infinitesimal calculus, and even in the subsequent period, people were much occupied with paradoxes of this character. A difficulty was found in the fact that the expression for a differential was never exact, save when the differential had become infinitely small,—a limit which could never be reached. The sum of non-infinitely small elements, it was thus thought, could give only an approximately correct result. It was sought to resolve this difficulty in all sorts of ways. But the actual practical uses to which the infinitesimal calculus is put are totally different from what is here assumed, as the simplest example will show, and are affected in no wise whatever by the imaginary difficulty in question.

If $y = x^m$, I find for an increment dx of x the increment

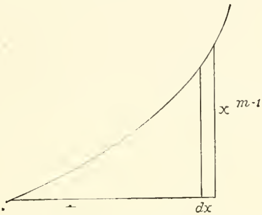
$$dy = mx^{m-1}\,dx + \frac{m(m-1)}{1\cdot 2}\,x^{m-2}\,dx^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\,x^{m-3}\,dx^3 + \ldots.$$

Having this result, it will be seen that the function x^m *reacts* in a definite manner in response to a definite operation, namely, that of

[1] Hankel, *Geschichte der Mathematik*, Leipsic, 1874, p. 149.

differentiation. This reaction is a characteristic mark of x^m, and stands on precisely the same footing as the bluish-green coloring which arises from dissolving copper in sulphuric acid. The number of terms that remain standing in the series is in itself indifferent. But the reaction is simplified by taking dx so small that the subsequent terms vanish with respect to the first. It is on account of this simplification only that dx is considered very small.

In a curve with the ordinate $z = mx^{m-1}$, it is seen that on increasing x by dx, the quadrature of the curve is increased by a small amount of surface, the expression for which when dx is very small



is simplified by reduction to the form $mx^{m-1}\,dx$. In response to the same operation as before, and under the same simplifying circumstances, the quadrature reacts as the familiar function x^m reacts. We recognise the function, thus, by its reaction.

If the mode in which the quadrature reacted did not accord with the mode of reaction of any function known to us, the entire method would leave us in the lurch. We should then have to resort to mechanical quadratures; we should actually be compelled to put up with finite elements; we should have to sum up finite numbers of these elements; and in such an event the result would be really inexact.

The twofold *salto mortale* from the finite to the infinitely small, and back again from this to the finite, is accordingly nowhere actually performed; on the contrary, the situation here is quite similar to that in every other domain of research. Acquaintance with mathematical and geometrical facts is acquired by actual employment with those facts. These, on making their appearance again, are recognised, and when they appear in part only, they are completed in thought, in so far as they are uniquely determined.[1]

The manner in which the conception of a continuum has arisen

[1] It is well known that differentials may be avoided by operating with differential coefficients which are the limiting values of the difference-quotients. Timid minds which find solace in this mode of conception will be content to put up with the cumbrousness sometimes involved.

will now be clear. In a sensory system the parts of which exhibit fluxional characteristics not readily admitting of distinction, we cannot retain the single parts either in the senses or in the imagination with any certainty. To be able to recognise definitely, therefore, the relations obtaining between the parts of such systems, we have to employ artificial devices such as measures. The mode of action of the measures is then substituted for the mode of action of the senses. Immediate contact with the system is lost by this procedure; and, furthermore, since the technology of measurement is founded on the technology of counting, numbers are substituted for the measures precisely as the measures were substituted for direct sense-perception. After we have once performed the operation of dividing a unit into component parts, and after we have once noticed that the parts exhibit the same properties as the original unit, then no obstacle presents itself to our continuing in thought to infinity the subdivision of the number which stands for the measure. But in doing so we imagine that we have also divided both the measure and system that is measured, into infinity. And this leads us to the notion of a continuum having the properties which we specified at the beginning of this article.

But it is not permissible to assume that everything that can be done with a sign or a number can also be done with the thing designated by that sign or number. Admitting that the number which is employed to specify a distance can be divided into infinity without any possibility whatever of meeting with obstacles, still the possibility of such division by no means necessarily applies to the distance itself. There is nothing that presents the *appearance* of a continuum but may still be composed of discrete elements, provided only those elements be sufficiently small as compared with our smallest practically applicable measures, or provided only they be sufficiently numerous.

Wherever we imagine we discover a continuum, all we can say is, that we can institute the same observations with respect to the smallest observable parts of the system in question as we can in the case of larger systems, and that we observe that the behavior of those parts is quite similar to that of the parts of larger systems. The length to which these observations may be carried can be decided by experience only. Where experience raises no protest, we may hold fast to the notion of a continuum, which is in no wise injurious and represents a convenient fiction only.